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Solvability of Multivariate Interpolation by Radial or Related Functions

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Let X be a linear space, and H a Hilbert space. Let $\mathcal{N}$ denote a set of n distinct points in X designated by $x_1, ..., x_n$ (these points are called *nodes*). It is desired to interpolate arbitrary data on $\mathcal{N}$ by a function in the linear span of the n functions,

$$x \mapsto \sum_{v=1}^{m} F_v(\|T_v(x-y_k)\|^2), \qquad k=1, ..., n,$$

where y_k are n distinct points in X (called *knots*), T_v are linear maps from X to H, and F_v are some suitable univariate functions. In this paper, we discuss the solvability of this interpolation scheme. For the case in which the nodes and knots coincide, we give a convenient condition which is equivalent to the nonsingularity of the interpolation matrices. We obtain some sufficient conditions for the case in which the nodes and knots do not necessarily coincide.

1. INTRODUCTION

Let X be a linear space, and H a Hilbert space. Let L denote the set of all linear maps from X to H. Given $g \in C(\mathbb{R})$ and $T \in L$, we can construct a simple-structured function Φ on X by composing g with $\|T\|$, where $\|\cdot\|$ is the norm on H; i.e., $\Phi(x) = g(\|T(x)\|)$, $x \in X$. If $\phi \in X^*$, the conjugate space of X, then $\Psi = g \circ \phi$ is also a function on X, and we call Ψ a *ridge* function on X. For example, the mapping $x \mapsto g(\langle x, v\rangle)$, where $x, v \in \mathbb{R}^d$ with v being fixed, and $\langle x, v\rangle$ being the Euclidean dot product of x and v, is by definition a ridge function on $\mathbb{R}^d$. It is constant on every line $\{a + tu : t \in \mathbb{R}\}$, where $a, u \in \mathbb{R}^d$ and $u \perp v$. If X is a Banach space, then it is proved by Cheney and Sun [SC] that the set of ridge functions is fundamental in the space $C(X)$ under the topology of compact convergence.

Let $\mathcal{N}$ denote a set of n distinct points in X designated by $x_1, ..., x_n$. These points are called *nodes*. Let some arbitrary data (x_j, f_j), $f_j \in \mathbb{R}$, be

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given on $\mathcal{N}$. We wish to interpolate the data by a function in the linear space generated by the n functions

$$x \mapsto \sum_{\nu=1}^{m} F_\nu(\|T_\nu(x-y_k)\|^2), \qquad k=1, ..., n, \tag{1}$$

where $y_1, ..., y_n$ are n distinct points in X (called *knots*), $T_1, ..., T_m \in L$, and $F_1, ..., F_m$ are some suitable functions in $C(\mathbb{R})$ that we will describe later. By varying the number m, the functions $F_1, ..., F_m$ and the linear maps $T_1, ..., T_m$, we obtain a rich family of interpolating functions which are of simple structure. For instance, included in (1) are sums of radial functions, ridge functions, or both.

When the interpolation conditions are imposed on an element of this linear space, the result is a system of n linear equations in the unknown coefficients $c_1, ..., c_n$,

$$\sum_{j=1}^{n} c_j \sum_{\nu=1}^{m} F_\nu(\|T_\nu(x_j-y_k)\|^2)=f_i, \qquad k=1, ..., n.$$

The coefficient matrix A of the linear system has entries

$$A_{jk}=\sum_{\nu=1}^{m} F_\nu(\|T_\nu(x_j-y_k\|^2), \tag{2}$$

and is termed the *interpolation matrix*. We also need to impose some conditions on the functions $F_1, ..., F_m$. Let $\mathscr{CM}$ denote the set of functions satisfying the following three conditions:

(C1) F: $[0, \infty) \to [0, \infty)$,

(C2) F is completely monotone on $(0, \infty)$ and continuous at 0,

(C3) F is not a constant.

Let $\mathscr{DM}$ denote the set of functions satisfying the following four conditions:

(D1) F: $[0, \infty) \to [0, \infty)$,

(D2) F is C^∞ on $(0, \infty)$ and continuous at 0,

(D3) F' is not a constant,

(D4) $(-1)^\nu F^{(\nu+1)}(t) \geqslant 0$ for $\nu=0, 1, 2, ...,$ and $t>0$.

We recall that a function f: $[0, \infty) \to \mathbb{R}$ is said to be completely monotone on $(0, \infty)$ if $(-1)^\nu f^{(\nu)}(x) \geqslant 0$, $\nu=0, 1, 2, ...,$ on $(0, \infty)$. Hence, condition (D4) is equivalent to F' being completely monotone on $(0, \infty)$. In [D, DLC, M], an extra condition was imposed on the functions in

$\mathscr{DM}$, namely, $F(t)>0$ when $t>0$. It was pointed out to the author by the referee that this condition is not essential in the argument.

Throughout this paper, if not declared otherwise, the functions F_ν will be assumed to be elements of $\mathscr{CM}$ or $\mathscr{DM}$.

So far only the special case of this interpolation scheme where the nodes and knots coincide (i.e., $x_j = y_j$ for all j), has been studied in the literature. Micchelli [M] proved that if $m=1$ and if T_1 is the identity map on $\mathbb{R}^d$, then the interpolation matrix of (2) is nonsingular. Dyn, Light, and Cheney [DLC] proved that if $m=d$, and $T_1, ..., T_d$ are projections from $\mathbb{R}^d$ to the coordinate axes, then the interpolation matrix of (2) is nonsingular if and only if the n functions in (1) are linearly independent. It was showed by Dyn and Micchelli [DM] that the same is true for the general functions in (1) and for a wider class of interpolation problems based on conditionally positive definite functions $\{F_\nu\}_{\nu=1}^m$ of arbitrary order.

Dyn, Light, and Cheney took a geometric approach. They used the concept of a *path* in $\mathbb{R}^2$. A path in $\mathbb{R}^2$ is an ordered set of points $P_1, P_2, ..., P_l$ in $\mathbb{R}^2$ such that the line segments $\overline{P_1P_2}, \overline{P_2P_3}, ...,$ are all of positive length and are alternatively horizontal and vertical. The path is *closed* if $P_l = P_1$, and if l is odd. In [DLC], it was shown that the interpolation matrix A, $A_{jk} = F_1(\|T_1(x_j - x_k)\|^2) + F_2(\|T_2(x_j - x_k\|^2)$, where $x_1, ..., x_n \in \mathbb{R}^2$ and T_1, T_2 are the two coordinate projections, is singular if and only if the set of nodes contains a closed path. The *closed path* introduced in [DLC] essentially works in a much more general setting in the case $m=2$ [DM]. But when $m \geqslant 3$, it seems that there is no analogy of the closed path property; see [LC]. In discussing the singularities of l_1-norm matrices, Light [L] introduced *multidimensional closed path*, using the structure of a tree. Light showed that if the set of nodes contains a multidimensional closed path, then the corresponding l_1-norm matrix is singular. On the other hand, there exist some node sets which have a singular matrix without containing a multidimensional closed path. Such an example will be given in Section 2.

Inspired by the work of Dyn and Micchelli [DM], we find that it is convenient to use a semi-norm to study the solvability of the interpolation problem. This semi-norm is closely related to the pattern of node distribution and the linear maps $T_1, ..., T_m$. We introduce and exploit this semi-norm in Section 2. Section 3 of this paper is devoted to the more general case where the nodes and knots do not necessarily coincide. We prove some sufficient conditions. However, our results there are confined to ridge functions.

2. The Semi-Norm

As in Section 1, a set of nodes (in X)

$$\mathscr{N} := \{x_1, ..., x_n\}$$

and a collection of linear maps (from X to H)

$$\mathcal{T} := \{T_1, ..., T_m\}$$

are given. For each ν, $\nu = 1, ..., m$, let $A_\nu := \{y : T_\nu x_j = y \text{ for some } j\}$. In other words, $A_\nu = T_\nu(\mathcal{N})$. We define the semi-norm $|\cdot|_{\mathcal{N},\mathcal{T}}$ on $\mathbb{R}^n$ associated with $\mathcal{N}$ and $\mathcal{T}$ in the following way: if $c := (c_1, ..., c_n) \in \mathbb{R}^n$, then

$$|c|_{\mathcal{N},\mathcal{T}} := \left[\sum_{\nu=1}^{m} \sum_{y \in A_\nu} \left(\sum \{c_j : T_\nu x_j = y\}\right)^2\right]^{1/2}. \tag{3}$$

It is easy to check that $|\cdot|_{\mathcal{N},\mathcal{T}}$ is indeed a semi-norm on $\mathbb{R}^n$. In this paper, our principal concern is to understand the precise conditions under which $|\cdot|_{\mathcal{N},\mathcal{T}}$ is a genuine norm and not just a semi-norm. Thus any other equivalent definition is also valid. For example, we can define the semi-norm as

$$\left[\sum_{\nu=1}^{m} \sum_{y \in A_\nu} \left|\sum \{c_j : T_\nu x_j = y\}\right|^p\right]^{1/p}, \qquad 1 \leq p < \infty.$$

We settle upon Eq. (3) as our definition for convenience.

We offer a few examples before we exploit properties of the semi-norm.

EXAMPLE 1. Let $m = 1$, and let T_1 be a linear map from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that $T(x_j) \neq T(x_k)$ if $j \neq k$. Then $|c|_{\mathcal{N},\mathcal{T}} = (\sum_{j=1}^n c_j^2)^{1/2}$, i.e., $|\cdot|_{\mathcal{N},\mathcal{T}}$ is the Euclidean norm on $\mathbb{R}^n$.

EXAMPLE 2. Let x_1, x_2, x_3, x_4 be the four vertices of a rectangle in the plane $\mathbb{R}^2$ (see Fig. 1(a)), and let T_1, T_2 be the two coordinate projections. Then (3) defines a semi-norm on $\mathbb{R}^4$ in the following way: for $c := (c_1, c_2, c_3, c_4) \in \mathbb{R}^4$,

$$|c|_{\mathcal{N},\mathcal{T}} = [(c_1 + c_2)^2 + (c_3 + c_4)^2 + (c_2 + c_3)^2 + (c_4 + c_1)^2]^{1/2}.$$

If we enlarge or shrink the rectangle, we always obtain the same semi-norm in this process. Actually, $|\cdot|_{\mathcal{N},\mathcal{T}}$ is a certain geometric invariant

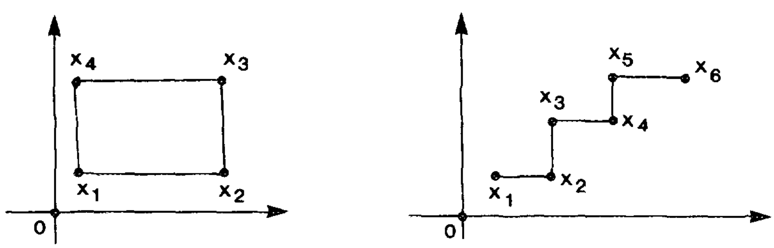

FIGURE 1

which describes some topological relationship between the node set $\mathcal{N}$ and $\mathcal{T}$. We also note that the semi-norm in this example is not a norm, since for the nonzero vector $c = (1, -1, 1, -1)$, we have $|c|_{\mathcal{N}, \mathcal{T}} = 0$.

EXAMPLE 3. Again in $\mathbb{R}^2$, we select six points as illustrated in Fig. 1(b). T_1, T_2 are projections to the coordinate axes. The result is the following semi-norm in $\mathbb{R}^6$,

$$|c|_{\mathcal{N}, \mathcal{T}} = [(c_1 + c_2)^2 + (c_3 + c_4)^2 + (c_5 + c_6)^2 + c_1^2 + (c_2 + c_3)^2 + (c_4 + c_5)^2 + c_6^2]^{1/2}.$$

This is a norm since $|c|_{\mathcal{N}, \mathcal{T}} = 0$ is equivalent to the homogeneous linear system

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_3 + c_4 &= 0 \\ c_5 + c_6 &= 0 \\ c_1 &= 0 \\ c_2 + c_3 &= 0 \\ c_4 + c_5 &= 0 \\ c_6 &= 0 \end{aligned}$$

which has only the trivial solution.

EXAMPLE 4. In $\mathbb{R}^3$, we select five vertices of the unit cube as illustrated in Fig. 2. Let T_1, T_2, T_3 be the three coordinate projections. Then we have a semi-norm on $\mathbb{R}^5$ defined by

$$|c|_{\mathcal{N}, \mathcal{T}} = [(c_1 + c_3 + c_4)^2 + (c_2 + c_5)^2 + (c_1 + c_2 + c_4)^2 + (c_3 + c_5)^2 + (c_1 + c_2 + c_3)^2 + (c_4 + c_5)^2]^{1/2}.$$

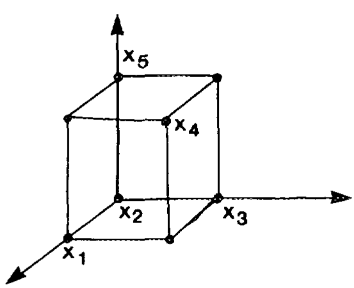

FIGURE 2

This is not a norm, as the nonzero vector $(2,-1,-1,-1,1)$ has a semi-norm 0.

These five points also serve as the example we mentioned in Section 1. Namely, they do not form a multidimensional closed path under the definition of Light [L], but their l_1 norm distance matrix is singular.

To verify that $|\cdot|_{\mathcal{N},\mathcal{T}}$ is a norm, we write down a homogeneous linear system whose coefficient matrix is sparse and all the nonzero entries of the matrix are 1. The semi-norm $|\cdot|_{\mathcal{N},\mathcal{T}}$ is a norm if and only if the linear system has only the trivial solution.

If $|\cdot|_{\mathcal{N},\mathcal{T}}$ is a norm on $\mathbb{R}^n$, then it is equivalent to the Euclidean norm on $\mathbb{R}^n$, as any two norms on a finite-dimensional Banach space are equivalent to each other. Let $I_{\mathcal{N},\mathcal{T}}$ be the identity operator from the Banach space $(\mathbb{R}^n, |\cdot|_{\mathcal{N},\mathcal{T}})$ to the Banach space $(\mathbb{R}^n, \|\cdot\|)$. Let the operator norm of $I_{\mathcal{N},\mathcal{T}}$ be $1/\lambda_{\mathcal{N},\mathcal{T}}$. Then, the number $\lambda_{\mathcal{N},\mathcal{T}}$ has the following properties:

(i) $|c|_{\mathcal{N},\mathcal{T}} \geqslant \lambda_{\mathcal{N},\mathcal{T}} \|c\|$ for all c in $\mathbb{R}^n$;

(ii) If λ is a positive number such that $|c|_{\mathcal{N},\mathcal{T}} \geqslant \lambda \|c\|$, then $\lambda_{\mathcal{N},\mathcal{T}} \geqslant \lambda$.

We call $\lambda_{\mathcal{N},\mathcal{T}}$ the norm constant of $|\cdot|_{\mathcal{N},\mathcal{T}}$ related to $\|\cdot\|$, or simply the norm constant of $|\cdot|_{\mathcal{N},\mathcal{T}}$.

PROPOSITION 1. *Let $\mathcal{N}$ and $\mathcal{T}$ be given, $\#\mathcal{N}=n$, and let $\mathcal{T}_1 \subset \mathcal{T}$. Then for any $c \in \mathbb{R}^n$, we have $|c|_{\mathcal{N},\mathcal{T}} \geqslant |c|_{\mathcal{N},\mathcal{T}_1}$. Consequently, if $|\cdot|_{\mathcal{N},\mathcal{T}_1}$ is a norm, so is $|\cdot|_{\mathcal{N},\mathcal{T}}$.*

Proof. From the definition of $|\cdot|_{\mathcal{N},\mathcal{T}}$ in Eq. (3), we have

$$|c|^2_{\mathcal{N},\mathcal{T}} = |c|^2_{\mathcal{N},\mathcal{T}\setminus\mathcal{T}_1} + |c|^2_{\mathcal{N},\mathcal{T}_1}.$$

So the result follows. ∎

PROPOSITION 2. *Let $\mathcal{T} := \{T_1, ..., T_m\}$ with $\ker(T_v) \neq \{0\}$ for all v. Then there exists a node set $\mathcal{N} := \{x_1, ..., x_{2^m}\}$, $\#\mathcal{N} = 2^m$, such that $|\cdot|_{\mathcal{N},\mathcal{T}}$ is not a norm on $\mathbb{R}^{2^m}$.*

Proof. The construction of $\mathcal{N}$ uses induction on m. When $m=1$, since $\ker(T_1) \neq \{0\}$, we can select two different points x_1, x_2 in X, so that $T(x_1)=T(x_2)$. Then for the nonzero vector $(1,-1) \in \mathbb{R}^2$, we have $|(1,-1)|_{\{x_1,x_2\},\{T_1\}} = 0$. Thus $|\cdot|_{\{x_1,x_2\},\{T_1\}}$ is not a norm on $\mathbb{R}^2$.

Suppose that the results is true for $m-1$. Then for $\mathcal{T}_1 := \{T_1, ..., T_{m-1}\}$, there exist a node set $\mathcal{N}_1 := \{x_1, ..., x_{2^{m-1}}\}$, $\#\mathcal{N}_1 = 2^{m-1}$, and a nonzero vector $u := (u_1, ..., u_{2^{m-1}})$ such that $|u|_{\mathcal{N}_1,\mathcal{T}_1} = 0$. i.e., for each v and every $y \in A_v$, $\sum \{u_j : T_v x_j = y\} = 0$. Since $\ker(T_m) \neq \{0\}$, we can select $z \neq 0$ such that $T_m(z)=0$ and such that $x_j \neq x_k - z$ for all j, $k = 1, ..., 2^{m-1}$. This is possible because the set $\{x_j - x_k : j, k = 1, ..., 2^{m-1}\}$ is finite. Let $\mathcal{N} := \{y_1, ..., y_{2^m}\} := \{x_1, ..., x_{2^{m-1}}, x_1 - z, ..., x_{2^{m-1}} - z\}$. Then $\# \mathcal{N} = 2^m$. Let $v := (v_1, ..., v_{2^m}) := (u_1, ..., u_{2^{m-1}}, -u_1, ..., -u_{2^{m-1}})$. We have $|v|^2_{\mathcal{N}, \mathcal{T}} = |v|^2_{\mathcal{N}, \mathcal{T}_1} + |v|^2_{\mathcal{N}, \{\mathcal{T}_m\}}$. It is clear that $|v|^2_{\mathcal{N}, \{\mathcal{T}_m\}} = 0$. It order to see that $|v|^2_{\mathcal{N}, \mathcal{T}_1} = 0$, we write down, for each $v < m$ and every $y \in A_v$,

$$\sum \{v_j : T_v y_j = y\} = \sum \{u_j : T_v x_j = y\} + \sum \{-u_j : T_v(x_j - z) = y\}$$

$$= \sum \{u_j : T_v x_j = y\} - \sum \{u_j : T_v x_j = T_v z + y\} = 0. \quad \blacksquare$$

PROPOSITION 3. *Given any node set* $\mathcal{N} := \{x_1, ..., x_n\}$, $\# \mathcal{N} = n$, *and* $m \geq 1$, *we can find a set of m elements* $\mathcal{T} := \{\psi_1, ..., \psi_m\} \subset X^*$, *such that* $|\cdot|_{\mathcal{N}, \mathcal{T}}$ *is a norm on* $\mathbb{R}^n$.

Proof. Select $\psi_1 \in X^*$ so that $\psi(x_j) \neq \psi(x_k)$ if $j \neq k$. This is possible because each set $\{\psi : \psi(x_j - x_k) = 0\}$ $(j \neq k)$ is a hyperplane in X^*, and their union is not all of X^*. Thus $|\cdot|_{\mathcal{N}, \{\psi_1\}}$ is a norm. Choose $\psi_2, ..., \psi_m$ arbitrarily and let $\mathcal{T} := \{\psi_1, ..., \psi_m\}$. By Proposition 1, $|\cdot|_{\mathcal{N}, \mathcal{T}}$ is also a norm. $\blacksquare$

If $m = 2$, the semi-norm $|\cdot|_{\mathcal{N}, \mathcal{T}}$ serves our purposes just as well as the "closed path" does. The following definition of *closed path* is given by Dyn and Micchelli [DM] which extends the one in [DLC].

DEFINITION 4. Let $\mathcal{T} := \{T_1, T_2\}$, where T_1, T_2 are two linear maps from X to H such that $\mathcal{T}$ is linearly independent. *A closed path* in X with respect to $\mathcal{T}$ is a finite ordered set consisting of l, l even, distinct points in X, $[y_1, ..., y_l]$, satisfying the following equalities

$$T_1 y_{2j-1} = T_1 y_{2j} T_2 y_{2j} = T_2 y_{2j+1}, \qquad j = 1, ..., l/2,$$

where $y_{l+1} = y_1$.

DEFINITION 5. Let $\mathcal{N}$ and $\mathcal{T}$ be given. If for every pair x_j and x_k in $\mathcal{N}$, $\mathcal{T}_v x_j = T_v x_k$ for all v implies $x_j = x_k$, then we say that $\mathcal{N}$ is in general position with respect to $\mathcal{T}$.

If $[y_1, ..., y_l]$ is a closed path and in general position with respect to $\mathcal{T} := \{T_1, T_2\}$, then we have

$$\begin{aligned} &T_1 y_{2j-1} = T_1 y_{2j}, \qquad T_2 y_{2j} = T_2 y_{2j+1}, \qquad j = 1, ..., l/2, \\ &T_2 y_{2j-1} \neq T_2 y_{2j}, \qquad T_1 y_{2j} \neq T_1 y_{2j+1}, \qquad j = 1, ..., l/2. \end{aligned} \tag{4}$$

This fact is crucial in the proof of Lemma 6 below.

LEMMA 6. *Let $\mathcal{N}$ be in general position with respect to $\mathcal{T} := \{T_1, T_2\}$. Let $\mathcal{M}$ be a nonvoid subset of $\mathcal{N}$ such that for any $y \in X$ and $v = 1, 2$, $\#(\mathcal{M} \cap \{x : T_v x = y\}) \neq 1$. Then $\mathcal{M}$ contains a closed path with respect to $\mathcal{T} = \{T_1, T_2\}$.*

Lemma 6 is a generalization of Lemma 3.2 in [DLC]. Essentially, the proof there works for Lemma 6 (use the remark following Definition 5). So we refer to [DLC] for the proof. ▌

THEOREM 7. *Let $\mathcal{N}$ be in general position with respect to $\mathcal{T} =: \{T_1, T_2\}$. In order that $|\cdot|_{\mathcal{N}, \mathcal{T}}$ be a norm it is necessary and sufficient that $\mathcal{N}$ contain no closed path with respect to $\mathcal{T}$.*

Proof. Assume that $\mathcal{N}$ contains a closed path. Remember the nodes if necessary so that $\{x_1, ..., x_m\}$ is a closed path, where $m \leqslant n$, m even. We show that for the nonzero vector

$$c := (\underbrace{1, -1, ..., 1, -1}_{m}, 0, ..., 0),$$

$|c|_{\mathcal{N}, \mathcal{T}} = 0$. For any $y \in A_1$, let $l_y = \#\{x_j : T_1 x_j = y, j = 1, ..., m\}$. Since $\mathcal{N}$ is in general position with respect to $\mathcal{T} =: \{T_1, T_2\}$, by Eq. (4), we see that l_y is even, and we have

$$\sum \{c_j : T_1 x_j = y\} = \underbrace{(1-1) + \cdots + (1-1)}_{l_y} = 0.$$

The same argument can be applied to $y \in A_2$. Therefore

$$|c|_{\mathcal{N}, \mathcal{T}} = \left[\sum_{v=1}^{2} \sum_{y \in A_v} \left(\sum \{c_j : T_v x_j = y\} \right)^2 \right]^{1/2} = 0.$$

The necessity of the theorem is proved.

To prove the sufficiency, assume that $\mathcal{N}$ does not contain a closed path. We have to show that the equation $|c|_{\mathcal{N}, \mathcal{T}} = 0$ implies that $c_j = 0$ for all j. We will prove this by induction on $\#\mathcal{N}$. When $\#\mathcal{N} = 1$, the result is automatically true. Assuming that the Theorem is true when $\#\mathcal{N} = n$, we show that it is also true when $\#\mathcal{N} = n + 1$. Since $\mathcal{N}$ does not contain a closed path, by Lemma 6, there exist an $x_{j_0} \in \mathcal{N}$ and a $T_{v_0} \in \mathcal{T}$ such that $\#(\mathcal{N} \cap (\{x : T_{v_0} x = T_{v_0} x_{j_0}\}) = 1$. So $T_{v_0} x_{j_0} \neq T_{v_0} x_j$ for all $j \neq j_0$. Then, by the definition of the semi-norm $|c|_{\mathcal{N}, \mathcal{T}}$, we know that $(|c|_{\mathcal{N}, \mathcal{T}})^2$ contains the term $c_{j_0}^2$. Hence $|c|_{\mathcal{N}, \mathcal{T}} = 0$ implies $c_{j_0} = 0$. Therefore, $|c|_{\mathcal{N}, \mathcal{T}} = |c'|_{\mathcal{N} \setminus \{x_{j_0}\}, \mathcal{T}} = 0$, where

$$c' := (\underbrace{c_1, ..., c_{j_0 - 1}, c_{j_0 + 1}, ..., c_{n+1}}_{n}).$$

Since $\#\{\mathcal{N}\backslash\{x_{j_0}\}\}=n$, the induction hypothesis implies that $c'=0$. Hence $c=0$. ∎

Let $\mathbb{R}^H$ and $\mathbb{R}^X$ denote the sets of all real-valued functions on H and X, respectively. Let $R(\mathcal{T}):=\operatorname{span}\{f\circ T_v : f\in\mathbb{R}^H,\ \ v=1,...,m\}$. The next theorem is about the relationship between the semi-norm $|\cdot|_{\mathcal{N},\mathcal{T}}$ and $R(\mathcal{T})$.

THEOREM 8. *The following three statements are equivalent*:

(i) $|\cdot|_{\mathcal{N},\mathcal{T}}$ *is not a norm*;

(ii) *There is a nontrivial functional* $\phi\in(\mathbb{R}^X)^*$ *supported on* $\mathcal{N}$ *which annihilates* $R(\mathcal{T})$;

(iii) *There exist n constants* $\alpha_1,...,\alpha_n$, *not all* α_j *are zero, such that*

$$\sum_{j=1}^{n}\alpha_j G(x-x_j)=0$$

for all $G\in R(\mathcal{T})$ *and all* $x\in\mathbb{R}^d$.

Proof. Let $x\in X$, and let x^* be the point evaluation functional associated with the point x; i.e., for any function g whose domain includes x, $x^*(g)=g(x)$. Let $\Phi(\mathcal{N})$ denote the set of all the linear functional $\phi\in(\mathbb{R}^X)^*$ with $\operatorname{supp}(\phi)\subset\mathcal{N}$. Then it is obvious that every element of $\Phi(\mathcal{N})$ can be represented as $\sum_{j=1}^{n}\alpha_j x_j^*$, $\alpha_1,...,\alpha_n\in\mathbb{R}$, and vice-versa.

For the proof that Part (ii) implies Part (i), suppose Part (ii) is true, and let ϕ be a nontrivial functional in $(\mathbb{R}^X)^*$, $\phi=\sum_{j=1}^{n}c_j x_j^*$, that annihilates $R(\mathcal{T})$. For each v, we have

$$\begin{aligned}0=\phi(f\circ T_v)&=\sum_{j=1}^{n}c_j f(T_v x_j)\\&=\sum_{y\in A_v}\sum\,[c_j f(T_v x_j): T_v x_j=y]=\sum_{y\in A_v}f(y)\sum\,[c_j: T_v x_j=y]=0.\end{aligned}\tag{5}$$

Since (5) is true for any function in $\mathbb{R}^H$, we must have

$$\sum\,[c_j: T_v x_j=y]=0\qquad (y\in A_v,\ v=1,...,m).\tag{6}$$

Equation (6) is equivalent to $|c|_{\mathcal{N},\mathcal{T}}=0$. Since ϕ is nontrivial, the vector c is not zero. This shows that $|\cdot|_{\mathcal{N},\mathcal{T}}$ is not a norm. Therefore part (i) is true.

In order to prove that part (i) implies part (ii), suppose that part (i) is true. Then there exists a nonzero vector $c:=(c_1,...,c_n)$ so that $|c|_{\mathcal{N},\mathcal{T}}=0$. From Eqs. (5) and (6), we see that the nontrivial functional $\phi:=\sum_{j=1}^{n}c_j x_j^*$ annihilates $R(\mathcal{T})$.

The equivalence between part (ii) and part (iii) follows from the observation that if $G(\cdot) \in R(\mathscr{T})$ then $G(x - \cdot) \in R(\mathscr{T})$ for any fixed $x \in X$. ∎

We point out here that the result of Theorem 8 is still true if we replace $R(\mathscr{T})$ by a certain subspace of it. Let $\mathscr{F}(H)$ be a subspace of $\mathbb{R}^H$ which separates the points of H, i.e., for any pair $x_1, x_2 \in H$, with $x_1 \neq x_2$, there exists a function $f \in \mathscr{F}(H)$ such that $f(x_1) \neq f(x_2)$. Let $\bar{R}(\mathscr{T})$ be a subspace of $R(\mathscr{T})$ defined by

$$\bar{R}(\mathscr{T}) := \text{span}\{f \circ T_v : f \in \mathscr{F}(H), v = 1, ..., m\}.$$

Then, Theorem 8 is true for $\bar{R}(\mathscr{T})$ with the same proof.

COROLLARY 9. *If $|\cdot|_{X,\mathscr{T}}$ is not a norm, then for any $G \in R(\mathscr{T})$, the interpolation matrix $(G(x_j - x_k))$ is singular.*

From the examples and theorems discussed above, we see that in some special cases $|\cdot|_{X,\mathscr{T}}$ being a norm is equivalent the nonsingularity of the interpolation matrix of (2) when $x_j = y_j$, $j = 1, ..., n$. The following theorem shows that this is true in general.

THEOREM 10. *Let the functions F_v, $v = 1, ..., m$, be either all from $\mathscr{CM}$ or $\mathscr{DM}$. Then the interpolation matrix A,*

$$A_{jk} = \sum_{v=1}^{m} F_v(\|T_v(x_j - x_k)\|^2),$$

is nonsingular if and only if the semi-norm $|\cdot|_{X,\mathscr{T}}$ is a norm.

Theorem 10 follows from [DM, Lemma 2.1 and Proposition 3.1]. A direct proof of Theorem 10 is also possible; see [S_2].

Now we have the following three equivalent conditions:

(i) The interpolation matrix of (2) when $x_j = y_j$, $j = 1, ..., n$, is nonsingular;

(ii) The n functions $x \mapsto \sum_{v=1}^{m} F_v(\|T_v(x - x_j)\|^2)$, $j = 1, ..., n$, are linearly independent;

(iii) The semi-norm $|\cdot|_{X,\mathscr{T}}$ is a norm.

The equivalence of (i) and (ii) was proved in special cases by Dyn, Light, and Cheney [DLC] and in general by Dyn and Micchelli [DM]. Practically, condition (iii) is very convenient. As we mentioned before, $|\cdot|_{X,\mathscr{T}}$ being a norm is equivalent to a certain linear system having only the trivial solution. The coefficient matrix of the system is sparse and all the nonzero entries are 1. Even if n is a reasonably big number, the problem poses no challenge to the modern computers.

If the interpolation matrix A,

$$A_{jk} = \sum_{\nu=1}^{m} F_\nu(\|T_\nu(x_j - x_k)\|^2),$$

is nonsingular, in which case $|\cdot|_{\mathcal{N}, \mathcal{T}}$ is a norm, then using the methods developed by Ball [B] and Narcowich and Ward [NW], it is possible to estimate the norm of A^{-1} (as an operator from l_2^n to l_2^n) by the norm constant $\lambda_{\mathcal{N}, \mathcal{T}}$ of $|\cdot|_{\mathcal{N}, \mathcal{T}}$. We will discuss the problem elsewhere.

At the end of this section, we prove some useful theorems using properties of the semi-norm and Theorem 10.

THEOREM 11. *Let $\mathcal{N}$ and $\mathcal{T}$ be given, and let $\mathcal{T}_1 \subset \mathcal{T}$. Let the functions $F_1, ..., F_m$ be either all from $\mathcal{CM}$ or all from $\mathcal{DM}$. Then, if the interpolation matrix B,*

$$B_{jk} = \sum_{T_\nu \in \mathcal{T}_1} F_\nu(\|T_\nu(x_j - x_k)\|^2)\},$$

is nonsingular, so is the matrix A

$$A_{jk} = \sum_{\nu=1}^{m} F_\nu(\|T_\nu(x_j - x_k)\|^2).$$

Theorem 11 shows that the interpolation scheme has the advantage that appropriate terms can be added without disturbing the solvability.

Proof. If the matrix B is nonsingular, then the semi-norm $|\cdot|_{\mathcal{N}, \mathcal{T}_1}$ is a norm by Theorem 10. By Proposition 1, the semi-norm $|\cdot|_{\mathcal{N}, \mathcal{T}}$ is also a norm. By Theorem 10 again, the matrix A is nonsingular. ∎

THEOREM 12. *Let $m \geqslant 1$, and $F_1, ..., F_m$ be either all from $\mathcal{CM}$ or all from $\mathcal{DM}$. Then for any set $\mathcal{N} := \{x_1, ..., x_n\}$ of distinct nodes, there exists a set of m elements $\psi_1, ..., \psi_m \in X^*$, such that the interpolation matrix A,*

$$A_{jk} = \sum_{\nu=1}^{m} F_\nu((\psi_\nu(x_j - x_k))^2),$$

is nonsingular.

Proof. Apply Proposition 3 and Theorem 10. ∎

We also have the following negative result:

THEOREM 13. *Let $\mathcal{T} := \{T_1, ..., T_m\}$ with* $\ker(T_\nu) \neq \{0\}$ *for all ν. Then* *there exists a node set* $\mathcal{N} := \{x_1, \ldots, x_{2^m}\}$, $\#\mathcal{N} = 2^m$, *such that the matrix* A,

$$A_{jk} = \sum_{\nu=1}^{m} h_\nu(T_\nu(x_j - x_k)),$$

is singular, where $h_1, \ldots, h_m$ *are arbitrarily functions in* $\mathbb{R}^H$.

Proof. For the given $\mathcal{T}$, by Proposition 2, there exists a node set $\mathcal{N} := \{x_1, \ldots, x_{2^m}\}$, $\#\mathcal{N} = 2^m$, such that the semi-norm $|\cdot|_{\mathcal{N}, \mathcal{T}}$ is not a norm. Since the function $\sum_{\nu=1}^{m} h_\nu \circ T_\nu$ belongs to $R(\mathcal{T})$, the matrix A is singular by Corollary 9. ▮

3. The Case in which the Nodes and Knots Do not Necessarily Coincide

Definition 14. Let A be an $n \times n$ matrix. Let $A\binom{i_1, i_2, \ldots, i_p}{j_1, j_2, \ldots, j_p}$ denote the $p \times p$ minor of A obtained by retaining only the rows labelled $i_1, i_2, \ldots, i_p$ and the columns labelled $j_1, j_2, \ldots, j_p$. Here we assume $1 \leqslant p \leqslant n$ and

$$1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n, \qquad 1 \leqslant j_1 < j_2 < \cdots < j_p \leqslant n. \tag{8}$$

If for all $1 \leqslant p \leqslant n$, and all $i_1, i_2, \ldots, i_p$, $j_1, j_2, \ldots, j_p$ satisfying (8), we have

$$\det A \begin{pmatrix} i_1, i_2, \ldots, i_p \\ j_1, j_2, \ldots, j_p \end{pmatrix} \geqslant 0 \tag{9}$$

we say that A is *totally positive*. If the strict inequality holds in (9), we say that A is *strictly totally positive*. This definition is in harmony with the one in [K].

It follows from a well-known theorem that a symmetric totally positive matrix is nonnegative definite, and that a symmetric strictly totally positive matrix is positive definite. Here we adhere to traditional terminology and call a matrix A positive definite if $x^T A x > 0$ when $x \neq 0$. The term "non-negative definite" is used if $x^T A x \geqslant 0$ for all x.

Lemma 15. *Let* $a_1 < a_2 < \cdots < a_n$ *and* $b_1 < b_2 < \cdots < b_n$. *Put* $A_{jk} = (a_j - b_k)^2$. *If* A *is symmetric, then* $c^T A c \leqslant 0$ *for every vector* c *satisfying* $\sum_{j=1}^{n} c_j = 0$.

Proof. The following Laplace transform formula is well-known. See, for example, [AS, p. 1022]

$$\Gamma(\beta)\, s^{-\beta} = \int_0^\infty e^{-st} t^{\beta-1}\, dt \qquad (\beta > 0,\ s > 0).$$

Let $0 < \alpha < 1$ and $\beta = 1 - \alpha$. The preceding equation leads to

$$\Gamma(1-\alpha) \int_0^u s^{\alpha-1}\, ds = \int_0^u \int_0^\infty e^{-st} t^{-\alpha}\, dt\, ds \qquad (0 < \alpha < 1).$$

Tonelli's Theorem [R, p. 270] justifies the interchange of integerations on the right. We obtain

$$u^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty (1 - e^{-ut})\, t^{-\alpha-1}\, dt \qquad (0 < \alpha < 1,\ u \geq 0).$$

Let $c \in \mathbb{R}^n$ and $\sum_{j=1}^n c_j = 0$. Put $Q_\alpha = \sum_{j=1}^n \sum_{k=1}^n c_j c_k (a_j - b_k)^{2\alpha}$. Then

$$Q_\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty \sum_{j=1}^n \sum_{k=1}^n c_j c_k t^{-\alpha-1} \{1 - \exp[-t(a_j - b_k)^2]\}\, dt.$$

Since $\sum_{j=1}^n c_j = 0$,

$$Q_\alpha = \frac{-\alpha}{\Gamma(1-\alpha)} \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty \sum_{j=1}^n \sum_{k=1}^n c_j c_k t^{-\alpha-1} \exp[-t(a_j - b_k)^2]\, dt.$$

Since the kernel $e^{-(t-s)^2}$ is strictly totall positive [K, p. 88] and since the matrix A is symmetric, the matrix defined by

$$E_{jk}(t) = \exp[-t(a_j - b_k)^2]$$

is totally positive and positive definite for all $t > 0$. Hence $Q_\alpha \leq 0$. Letting $\alpha \uparrow 1$, we obtain $c^T A c \leq 0$. ∎

LEMMA 16. *The following results about functions in* $\mathscr{CM}$ *or* $\mathscr{DM}$ *are true on the interval* $(0, \infty)$:

(i) *If* $f \in \mathscr{CM}$, *then* $f^{(2l)}$ *is strictly decreasing, while* $f^{(2l+1)}$ *is strictly increasing*, $l = 0, 1, 2, \ldots$.

(ii) *If* $f \in \mathscr{DM}$, *then* $f^{(2l)}$ *is strictly increasing, while* $f^{(2l+1)}$ *is strictly decreasing*, $l = 0, 1, 2, \ldots$. *In particular*, $F(t) > 0$ *when* $t > 0$.

Proof. Let $F \in \mathscr{CM}$, and $l = 0$. By the Bernstein–Widder Theorem [W], there exists a finite positive measure $\beta(s)$ on $[0, \infty)$, which is not concentrated at 0, such that $f(t) = f(0) + \int_0^\infty e^{-st}\, d\beta(s)$. If $0 \leq t_1 < t_2$, then

$e^{-t_1 s} - e^{-t_2 s} > 0$ for all $s > 0$. Hence $f(t_1) > f(t_2)$. The strict inequality holds because the measure $\beta(s)$ is not concentrated at 0. The other cases can be handled similarly. ∎

LEMMA 17. *Let* $a_1 < a_2 < \cdots < a_n$ *and* $b_1 < b_2 < \cdots < b_n$, *and let* $f \in \mathscr{C}\mathscr{M}$. *Put* $A_{jk} = f((a_j - b_k)^2)$. *If* A *is symmetric, then it is positive definite.*

Proof. By the Bernstein–Widder Theorem [W], we can express $f(x)$ as follows.

$$f(t) = \int_0^\infty e^{-st}\, d\beta(s) \qquad (t \geq 0).$$

where $\beta(s)$ is a positive Borel measure on $[0, \infty)$ whose mass is not concentrated at 0.

We have $A_{jk} = \int_0^\infty \exp[-s(a_j - b_k)^2]\, d\beta(s)$. As in Lemma 15, the matrix $E(s)$ given by

$$E_{jk}(s) = \exp[-s(a_j - b_k)^2]$$

is strictly totally positive for $s > 0$. Since f is strictly decreasing (Lemma 16), the matrix having elements $(a_j - b_k)^2$ is symmetric. Thus $E(s)$ is symmetric, and therefore positive definite. It follows that A is positive definite, for if $c \in \mathbb{R}^n \setminus \{0\}$, then

$$c^T A c = \int_0^\infty \sum_{j=1}^n \sum_{k=1}^n c_j c_k E_{jk}(s)\, d\beta(s) > 0.$$

Here again we use the fact that the measure $d\beta$ is not concentrated at 0. ∎

LEMMA 18. *Let* $a_1 < a_2 < \cdots < a_n$ *and* $b_1 < b_2 < \cdots < b_n$, *and let* $f \in \mathscr{D}\mathscr{M}$. *Put* $A_{jk} = f((a_j - b_k)^2)$. *If* A *is symmetric, then* A *has* $(n-1)$ *negative eigenvalues and* 1 *positive eigenvalue.*

Proof. By Lemma 3 in $[S_1]$, $f(x)$ can be expressed as

$$f(t) = f(0) + \int_0^\infty s^{-1}(1 - e^{-st})\, d\beta(s) \qquad (t \geq 0),$$

where $\beta(s)$ is a positive Borel measure on $[0, \infty)$ satisfying $\int_1^\infty (d\beta(s)/s) < \infty$ and $\int_{0+}^\infty d\beta(s) > 0$. The latter is equivalent to the fact that the mass of $\beta(s)$ is not concentrated at 0. Let α denote the mass of $\beta(s)$ at 0, i.e., $\alpha = \lim_{s \downarrow 0} [\beta(s) - \beta(0)]$. Then

$$f(t) = f(0) + \alpha t + \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty s^{-1}(1 - e^{-st})\, d\beta(s) \qquad (t \geq 0).$$

Let $c \in \mathbb{R}^n$, $c \neq 0$, and $\sum_{j=1}^n c_j = 0$. We have then

$$c^T A c = \sum_{j=1}^n \sum_{k=1}^n c_j c_k \Bigg[f(0) + \alpha(a_j - b_k)^2 + \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty s^{-1} [1 - \exp[-s(a_j - b_k)^2]] \, d\beta(s) \Bigg].$$

Since $\sum_{j=1}^n c_j = 0$, it follows that

$$\sum_{j=1}^n \sum_{k=1}^n c_j c_k \left[f(0) + \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty s^{-1} \, d\beta(s) \right] = 0.$$

By Lemma 15, $\sum_{j=1}^n \sum_{k=1}^n c_j c_k (a_j - b_k)^2 \leqslant 0$. Consequently,

$$c^T A c \leqslant -\lim_{\varepsilon \downarrow 0} \int_0^\infty \sum_{j=1}^n \sum_{k=1}^n c_j c_k s^{-1} E_{jk}(s) \, d\beta(s) < 0.$$

Here we use the facts that the function $s \mapsto s^{-1} E_{jk}(s)$ is positive for all $s > 0$ and that the measure $d\beta(s)$ is not concentrated at 0, (see the similar argument in Lemma 17). By the Courant–Fischer Theorem, the matrix A has at least $(n-1)$ negative eigenvalues. But the trace of A is nonnegative, hence A has exactly $(n-1)$ negative eigenvalues and 1 positive eigenvalue. ∎

A direct consequence of Lemmas 17 and 18 is

COROLLARY 19. *Let X be a linear space, and $\phi \in X^*$. Let $f \in C[0, \infty)$ and $F = f \circ \phi^2$. Let $x_1, ..., x_n$ and $y_1, ..., y_n$ be points in X such that $\phi(x_1) < \cdots < \phi(x_n)$ and $\phi(y_1) < \cdots < \phi(y_n)$. Assume that the matrix A, $A_{jk} = F(x_j - y_k)$ is symmetric. We have the following results:*

(1) *If $f \in \mathscr{CM}$, then A is positive definite.*

(2) *If $f \in \mathscr{DM}$, then A has $(n-1)$ negative eigenvalues and 1 positive eigenvalue.*

We give an example of a function F, a node set $\{x_1, ..., x_n\}$ and a knot set $\{y_1, ..., y_n\}$ distinct from the node set, such that the matrix $(F(x_j - x_k))$ is symmetric. In $\mathbb{R}$, let $x_j = j$, $y_j = (n-j) + \frac{1}{2}$, $j = 1, ..., n$, and let $F(t) = |t|$. It is obvious to see that the matrix $(F(x_j - x_k))$ is symmetric.

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